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Schrodinger Equations for Higher Order Non-relativistic Particles and N-Galilean Conformal Symmetry

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Abstract

We consider Schrödinger equations for a non-relativistic particle obeying N+1-th order higher derivative classical equation of motion. These equations are invariant under N(odd)-extended Galilean conformal (NGC) algebras in general d+1 dimensions. In 2+1 dimensions, the exotic Schrödinger equations are invariant under N (even)-GCA.

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I. INTRODUCTION

The Schrödinger equation for a non-relativistic free particle

$$(i\partial_t + \frac{1}{2M}\nabla^2)\psi(t,\mathbf{x}) = \mathbf{0} \tag{1}$$

is invariant under the scalar projective representation of the Galilei group [1][2]. In addition the maximal symmetry of (1) is the Schrödinger algebra [3][4] that includes extra generators associated with dilatation, expansion (special conformal transformations in one dimension) and a central charge. The Schrödinger equation is obtained by the canonical quantization of the non-relativistic free particle whose action $S = \int dt\, \frac{M}{2}(\frac{d\mathbf{X}}{dt})^2$ is also invariant under the Schrödinger group.

In this paper we will generalize above result (N=1 case) and show that the higher order non-relativistic particle model given by the Lagrangian[1]

$$\mathcal{L}_X = \frac{M}{2}\,(\frac{d^{\frac{N+1}{2}}\mathbf{X}}{dt^{\frac{N+1}{2}}})^2, \qquad N = 1, 3, 5, ..., \tag{2}$$

has the N-Galilean conformal (NGC) symmetry [7][8][2]. M is the constant with the dimension $[M] = (\text{mass})^{2-N}$. Furthermore the corresponding Schrödinger equations are projective representation of the NGC symmetries. For even N case we consider only 2+1 dimensions where the central extension of the NGC algebra exists.

The NGC algebra[7][8] is the finite dimensional extension of the Galilean algebra for positive integer N. Their commutators are

$$\begin{aligned}
&[D,H] = i\,H, \quad [C,H] = 2i\,D, \quad [D,C] = -i\,C,\\
&[H,\boldsymbol{\mathcal{P}}_j] = -i\,j\,\boldsymbol{\mathcal{P}}_{j-1}, \quad [D,\boldsymbol{\mathcal{P}}_j] = -i\,(j-\frac{N}{2})\,\boldsymbol{\mathcal{P}}_j,\\
&[C,\boldsymbol{\mathcal{P}}_j] = -i\,(j-N)\,\boldsymbol{\mathcal{P}}_{j+1}, \quad [\mathbf{J}_{ab},\boldsymbol{\mathcal{P}}_{j,c}] = i\,\delta_{c[b}\,\boldsymbol{\mathcal{P}}_{j,a]},\\
&[\mathbf{J}_{ab},\mathbf{J}_{cd}] = i\left(\delta_{c[b}\mathbf{J}_{a]d} - \delta_{d[b}\mathbf{J}_{a]c}\right).
\end{aligned} \tag{3}$$

It has subalgebras, (H,D,C), one dimensional conformal algebra and $(H,\boldsymbol{\mathcal{P}}_0,\boldsymbol{\mathcal{P}}_1,\mathbf{J})$, the unextended Galilean algebra and the acceleration extended Galilean algebra [6] with higher

[1] The N=3 case was considered in [5] as an example of a realization of the Galilei algebra with zero mass. The model was also considered in [6].

[2] In [9] it was conjectured that N+1-th order free equations of motion are NGC invariant for any N, odd and even.

order accelerations $\boldsymbol{\mathcal{P}}_j, (j = 2, ..., N)$. N is interpreted in terms of the dynamical exponent z under the dilation D of the coordinates as

$$t \to \lambda^z t, \qquad \mathbf{X} \to \lambda \mathbf{X}, \qquad z = \frac{2}{N}. \tag{4}$$

Recently it gets much attention on the Galilean conformal symmetry and its extensions especially in condensed matter physics and gravity [10] [11] [12] [9]. It is interesting to see how such symmetries are realized in a simple particle models.

In sect.2 we introduce a particle action invariant under central extension of the NGC algebra (3). In sect.3 we discuss it in quantum theory that how the symmetry is realized in the Schrödinger equation. Even N cases in $2+1$ dimensions are briefly commented in sect.4 and summary and discussions are in section 5. In appendix we add how the invariant actions are derived using the method of non-linear realization.

II. NGC INVARIANT PARTICLE MODEL

In order to quantize the particle model described by the Lagrangian (2) we construct the Hamiltonian associated with it. Since the theory is higher order in time derivative we introduce auxiliary coordinates $\mathbf{X}^j = \dot{\mathbf{X}}^{j-1}, \ (1 \leq j \leq \frac{N-1}{2}), \ \mathbf{X}^0 \equiv \mathbf{X}$ and the Lagrange multipliers $\mathbf{Y}_j$ to get the Lagrangian as

$$\mathcal{L}_X = \frac{M}{2}(\dot{\mathbf{X}}^{\frac{N-1}{2}})^2 + \sum_{j=0}^{\frac{N-3}{2}} \mathbf{Y}_j(\dot{\mathbf{X}}^j - \mathbf{X}^{j+1}). \tag{5}$$

The Ostrogradsky momenta [13] are

$$\mathbf{P}_j = \mathbf{Y}_j, \quad (j = 0, ..., \frac{N-3}{2}), \quad \mathbf{P}_{\frac{N-1}{2}} = M\,\dot{\mathbf{X}}^{\frac{N-1}{2}}. \tag{6}$$

If we use the second class constraint $(\mathbf{Y}_j, \mathbf{P}_Y^j) = (\mathbf{P}_j, 0)$, the independent canonical pairs are $(\mathbf{X}^j, \mathbf{P}_j), \ (0 \leq j \leq \frac{N-1}{2})$ and the Hamiltonian becomes

$$\mathcal{H} = \sum_{j=0}^{\frac{N-3}{2}} \mathbf{P}_j \mathbf{X}^{j+1} + \frac{1}{2M}\,(\mathbf{P}_{\frac{N-1}{2}})^2. \tag{7}$$

Using the canonical commutators

$$[\mathbf{X}^j, \mathbf{P}_k] = i\,\delta_{j,k}, \qquad [\mathbf{X}^j, \mathbf{X}^k] = [\mathbf{P}_j, \mathbf{P}_k] = 0, \tag{8}$$

the Heisenberg equations

$$\dot{\mathbf{X}}^j = \mathbf{X}^{j+1}, \qquad \dot{\mathbf{P}}_{j+1} = -\mathbf{P}_j, \qquad (0 \le j \le \frac{N-3}{2}),$$
$$\dot{\mathbf{X}}^{\frac{N-1}{2}} = \frac{1}{M}\mathbf{P}_{\frac{N-1}{2}}, \qquad \dot{\mathbf{P}}_0 = 0, \tag{9}$$

reproduce the Euler-Lagrangian equation of (2), $\frac{d^{N+1}}{dt^{N+1}}\mathbf{X}^0 = 0$. The Lagrangian (5) is invariant under the NGC symmetry whose hermitian canonical generators are

$$\begin{aligned}
H &= -\mathcal{H},\\
D &= -t\,\mathcal{H} + \sum_{j=0}^{\frac{N-1}{2}} (\frac{N}{2} - j)\frac{\{\mathbf{X}^j, \mathbf{P}_j\}_+}{2},\\
C &= -t^2\,\mathcal{H} + \sum_{j=0}^{\frac{N-1}{2}} \Big((N-2j)\,t\,\frac{\{\mathbf{X}^j, \mathbf{P}_j\}_+}{2}\\
&\qquad +(N-j+1)j\,\mathbf{X}^{j-1}\,\mathbf{P}_j\Big) - \frac{M}{2}\,(\frac{N+1}{2})^2(\mathbf{X}^{\frac{N-1}{2}})^2,\\
\boldsymbol{\mathcal{P}}_j &= j!\left(\sum_{\ell=0}^{j} \frac{t^{j-\ell}}{(j-\ell)!}\,\mathbf{P}_\ell\right), \qquad (j = 0, ..., \frac{N-1}{2}),\\
\boldsymbol{\mathcal{P}}_j &= j!\left(\sum_{\ell=0}^{\frac{N-1}{2}} \frac{t^{j-\ell}}{(j-\ell)!}\,\mathbf{P}_\ell - M\sum_{\ell=\frac{N+1}{2}}^{j} (-)^{\frac{N+1}{2}+\ell}\times\right.\\
&\qquad \left.\frac{t^{j-\ell}}{(j-\ell)!}\,\mathbf{X}^{N-\ell}\right), \qquad (j = \frac{N+1}{2}, ..., N),\\
\mathbf{J}_{ab} &= \sum_{j=0}^{\frac{N-1}{2}} (\mathbf{X}_b^j\,\mathbf{P}_{ja} - \mathbf{X}_a^j\,\mathbf{P}_{jb}).
\end{aligned} \tag{10}$$

where $\{\mathbf{X}^j, \mathbf{P}_k\}_+ = \mathbf{X}^j\mathbf{P}_k + \mathbf{P}_k\mathbf{X}^j$. These generators, G, that have explicit t dependence, are constant of motion, i.e., $\frac{d}{dt}G = -i[G, \mathcal{H}] + \frac{\partial G}{\partial t} = 0$. It proves the transformations generated by G's are symmetry of the Lagrangian (5). Transformations of $\mathbf{Y}_j$ are given by those of $\mathbf{P}_j$. Their commutators are those of the NGC algebra (3). In addition $\boldsymbol{\mathcal{P}}_j$'s are no longer commuting but appears a central charge $Z = M$ [14],

$$[\boldsymbol{\mathcal{P}}_j^a, \boldsymbol{\mathcal{P}}_k^b] = -i\,\delta^{ab}\,\delta^{N,j+k}\,(-1)^{\frac{k-j+1}{2}}k!\,j!\,Z. \tag{11}$$

In appendix we show that in a non-linear realization method [15] the Lagrangian (5) appears as the left invariant one-form associated with the central charge Z in (11).

III. SCHRODINGER EQUATION

Now we consider the Schrödinger equation associated with the Hamiltonian (7). The canonical pairs $(\mathbf{X}^j, \mathbf{P}_j)$ satisfying (8) are realized as the hermitian linear operators $(\mathbf{x}^j, -i\,\nabla_j)$ on complex wave functions $\psi(t, \mathbf{x}^0, ..., \mathbf{x}^{\frac{N-1}{2}})$ with the inner product

$$\langle\psi_1|\psi_2\rangle = \int d^d\mathbf{x}^0 \cdots d^d\mathbf{x}^{\frac{N-1}{2}}\ \overline{\psi_1(t, \mathbf{x}^j)}\, \psi_2(t, \mathbf{x}^j), \tag{12}$$

where $\overline{\psi}$ is the complex conjugate function of ψ. The Schrödinger equation $i\partial_t\psi = \mathcal{H}\psi$ for the wave function $\psi(t, \mathbf{x}^0, ..., \mathbf{x}^{\frac{N-1}{2}})$ is

$$\Phi_S\, \psi(t, \mathbf{x}^0, ..., \mathbf{x}^{\frac{N-1}{2}}) = 0. \tag{13}$$

where Φ_S is the Schrödinger differential operator defined by

$$\Phi_S = i\frac{\partial}{\partial t} - \mathcal{H} = i\frac{\partial}{\partial t} + \left(i\sum_{j=0}^{\frac{N-3}{2}} \mathbf{x}^{j+1}\nabla_j + \frac{1}{2M}\,(\nabla_{\frac{N-1}{2}})^2 \right). \tag{14}$$

The Schrödinger equation can be deduced from the action

$$I = \int dt\, d^d\mathbf{x}^j\, \overline{\psi(t, \mathbf{x}^j)}\, \Phi_S\, \psi(t, \mathbf{x}^j), \tag{15}$$

which is invariant under NGC transformations in the scalar projective representation of the wave function. The NGC algebra (3) is realized by the generators (10) in the form of hermitian vector fields on the wave functions $\psi(t, \mathbf{x}^1, ..., \mathbf{x}^{\frac{N-1}{2}})$,

$$\begin{aligned}
&H = -\mathcal{H},\\
&D = -t\,\mathcal{H} + \sum_{j=0}^{\frac{N-1}{2}} (\frac{N}{2} - j)\mathbf{x}^j(-i\nabla_j) - i\,d\frac{(N+1)^2}{16},\\
&C = -t^2\,\mathcal{H} + \sum_{j=0}^{\frac{N-1}{2}} \big((N-2j)t\,\mathbf{x}^j + (N-j+1)j\,\mathbf{x}^{j-1}\big)\\
&\qquad \times(-i\nabla_j) - \frac{M}{2}\,(\frac{N+1}{2})^2(\mathbf{x}^{\frac{N-1}{2}})^2 - i\,d\,\frac{(N+1)^2}{8}\,t,\\
&\boldsymbol{\mathcal{P}}_j = j!\left(\sum_{\ell=0}^{j} \frac{t^{j-\ell}}{(j-\ell)!}\,(-i\nabla_\ell)\right), \qquad (j = 0, ..., \frac{N-1}{2}),\\
&\boldsymbol{\mathcal{P}}_j = j!\left(\sum_{\ell=0}^{\frac{N-1}{2}} \frac{t^{j-\ell}}{(j-\ell)!}(-i\nabla_\ell) - M\sum_{\ell=\frac{N+1}{2}}^{j} (-)^{\frac{N+1}{2}+\ell}\right.\\
&\qquad \left.\times\,\frac{t^{j-\ell}}{(j-\ell)!}\,\mathbf{x}^{N-\ell}\right), \qquad (j = \frac{N+1}{2}, ..., N)\\
&Z = M,
\end{aligned} \tag{16}$$

where d is the spatial dimensions. They are satisfying the NGC algebra (3) and (11). These generators commute with the Φ_S, showing that they are constant of motion. Therefore the Schrödinger equation (13) remains invariant under the NGC transformations. When $\psi(t, \mathbf{x}^j)$ is a solution of the Schrödinger equation then $\psi'(t, \mathbf{x}^j) = e^{i\alpha G}\psi(t, \mathbf{x}^j)$ is also a solution, for $G = (H, D, C, \boldsymbol{\mathcal{P}}_j, \mathbf{J}_{ab})$. Since the generators are hermitian the transformations $U = e^{i\alpha G}$ are unitary. The transformed wave function $\psi'(t, \mathbf{x}^j)$ is also written in a form

$$\psi'(t, \mathbf{x}^j) = e^{i\alpha G}\psi(t, \mathbf{x}^j) = e^{\mathcal{A}+i\mathcal{B}}\psi(t', \mathbf{x}^{j'}), \tag{17}$$

using the Schrödinger equation. Here $\mathcal{A}, \mathcal{B}$ are real functions of the coordinates and the transformation parameters α's. $(t', \mathbf{x}^{j'})$ are the coordinates transformed by the NGC transformation.

The H transformation is time translation,

$$\psi'(t, \mathbf{x}^j) = e^{iaH}\psi(t, \mathbf{x}^j) = e^{a(\partial_t)}\psi(t, \mathbf{x}^j) = \psi(t + a, \mathbf{x}^j) \tag{18}$$

For the *finite* scale transformation,

$$\psi'(t, \mathbf{x}^j) = e^{i\lambda D}\psi(t, \mathbf{x}^j) = e^{\lambda\, d\, \frac{(N+1)^2}{16}}\, \psi(e^{\lambda}t, e^{\lambda(\frac{N}{2}-j)}\mathbf{x}^j), \tag{19}$$

under which the action (15) is invariant. For the *finite* conformal transformation, $\alpha G = \kappa C$,

$$\psi'(t, \mathbf{x}^j) = e^{i\kappa C}\psi(t, \mathbf{x}^j) = e^{\mathcal{A}+i\mathcal{B}}\psi(t', \mathbf{x}^{j'}). \tag{20}$$

with

$$\begin{aligned}
t' &= \frac{t}{1-\kappa t}, \quad \mathbf{x}^{j'} = \sum_{r=0}^{j} \frac{{}_jC_r\, r!\kappa^r\, {}_{N-j+r}C_r}{(1-\kappa t)^{N-2j+r}}\, \mathbf{x}^{j-r}, \\
e^{\mathcal{A}} &= (1-\kappa t)^{-\frac{d}{2}(\frac{N+1}{2})^2}, \quad \mathcal{B} = -\frac{\kappa M}{2}(\frac{N+1}{2})^2\, \frac{g_N}{(1-\kappa t)}, \\
g_N &\equiv \sum_{r,s=0}^{\frac{N-1}{2}} \frac{\gamma_r\gamma_s}{r!s!(r+s+1)}(\frac{\kappa}{1-\kappa t})^{r+s}\, \mathbf{x}^{\frac{N-1}{2}-r}\, \mathbf{x}^{\frac{N-1}{2}-s}, \\
&\quad \gamma_r \equiv {}_{\frac{N-1}{2}}C_{\frac{N-1}{2}-r}\, (r!)^2\, {}_{\frac{N+1}{2}+r}C_{\frac{N+1}{2}}.
\end{aligned} \tag{21}$$

For the *finite* $\boldsymbol{\mathcal{P}}_j$ transformations,

$$\psi'(t, \mathbf{x}^j) = e^{i\sum_{j=0}^{N}\boldsymbol{\beta}^j\boldsymbol{\mathcal{P}}_j}\psi(t, \mathbf{x}^j) = e^{-i2\pi\omega_1(t,\mathbf{x};\boldsymbol{\beta})}\, \psi(t, \mathbf{x}^{j'}). \tag{22}$$

The non-trivial projective phase [1] ω_1 associated with the $\boldsymbol{\mathcal{P}}_j$ transformations is given by

$$2\pi\omega_1(t,\mathbf{x};\boldsymbol{\beta}) = M\sum_{k=0}^{\frac{N-1}{2}} (-)^{\frac{N-1}{2}+k}(\mathbf{x}^k + \frac{\tilde{\boldsymbol{\beta}}^k}{2})\tilde{\boldsymbol{\beta}}^{N-k}, \tag{23}$$

where the transformed coordinates are given by

$$\mathbf{x}^{0\prime} = \mathbf{x}^0 + \tilde{\boldsymbol{\beta}}^0, \quad \mathbf{x}^{\ell\prime} = \mathbf{x}^\ell + \tilde{\beta}^\ell, \quad \tilde{\boldsymbol{\beta}}^\ell(t) \equiv \frac{d^\ell}{dt^\ell}\sum_{j=0}^{N} t^j \boldsymbol{\beta}^j. \tag{24}$$

We have a projective representation of the NGC group.

Under successive $\boldsymbol{\mathcal{P}}_j$ transformations we get the non-trivial 2-cocycle $\omega_2(\boldsymbol{\beta},\boldsymbol{\beta}')$,

$$U(\boldsymbol{\beta})U(\boldsymbol{\beta}') = e^{-i2\pi\omega_2(\boldsymbol{\beta},\boldsymbol{\beta}')}\,U(\boldsymbol{\beta}+\boldsymbol{\beta}'), \tag{25}$$

$$2\pi\,\omega_2(\boldsymbol{\beta},\boldsymbol{\beta}') = M\sum_{k=0}^{\frac{N-1}{2}} \frac{(-)^{\frac{N-1}{2}+k}}{2}\left(\tilde{\boldsymbol{\beta}}^k\,\tilde{\boldsymbol{\beta}}'^{N-k} - \tilde{\boldsymbol{\beta}}'^k\,\tilde{\boldsymbol{\beta}}^{N-k}\right). \tag{26}$$

The projective invariance of the Schrödinger equation is one to one correspondence with the fact that the higher order Lagrangian (2), or the corresponding Lagrangian (5) is invariant up to a total derivative under the *finite* $\boldsymbol{\mathcal{P}}_j$ transformations [16],

$$\mathcal{L}'_X = \mathcal{L}_X + \frac{d}{dt}(2\pi\,\omega_1), \tag{27}$$

where transformations of $\mathbf{X}^j$ and $\mathbf{Y}_j$ are ,

$$\mathbf{X}^{\ell\prime} = \mathbf{X}^\ell + \tilde{\boldsymbol{\beta}}^\ell, \quad \mathbf{Y}'_j = \mathbf{Y}_j + M(-)^{\frac{N-1}{2}+j}\tilde{\boldsymbol{\beta}}^{N-j}. \tag{28}$$

These two properties are related to the fact that NGC algebra has a central extension (11) [17].

IV. EVEN N MODEL IN 2+1 DIMENSIONS

In 2+1 dimensions we can construct a local higher order Lagrangian given by

$$\mathcal{L}_X = \frac{M}{2}\,\epsilon^{ab}\,\frac{d^{\frac{N}{2}}\mathbf{X}_a}{dt^{\frac{N}{2}}}\frac{d^{\frac{N}{2}+1}\mathbf{X}_b}{dt^{\frac{N}{2}+1}}, \tag{29}$$

where N is any positive even integer[3]. The Lagrangian equivalent to (29) is

$$\mathcal{L}_X = \frac{M}{2}\,\epsilon^{ab}\mathbf{X}_a^{\frac{N}{2}}\,\dot{\mathbf{X}}_b^{\frac{N}{2}} + \sum_{j=0}^{\frac{N}{2}-1}\mathbf{Y}_j(\dot{\mathbf{X}}^j - \mathbf{X}^{j+1}). \tag{30}$$

[3] The case N=2 was analyzed in [18].

The Ostrogradsky momenta [13] are

$$\mathbf{P}_j = \mathbf{Y}_j, \quad (j = 0, ..., \frac{N}{2} - 1), \qquad \mathbf{P}^b_{\frac{N}{2}} = \frac{M}{2}\,\mathbf{X}^{\frac{N}{2}}_a\,\epsilon^{ab}. \tag{31}$$

Using the second class constraints

$$\chi^a = \mathbf{P}^a_{\frac{N}{2}} + \frac{M}{2}\epsilon^{ab}\mathbf{X}^{\frac{N}{2}b} = 0, \qquad a = 1, 2, \tag{32}$$

the variables $\mathbf{X}^{\frac{N}{2}}_a, (a = 1, 2)$ can be expressed in terms of the independent $\mathbf{P}^a_{\frac{N}{2}}, (a = 1, 2)$ satisfying

$$\left[\mathbf{P}^a_{\frac{N}{2}}, \mathbf{P}^b_{\frac{N}{2}}\right] = -i\frac{M}{4}\epsilon^{ab}. \tag{33}$$

Note that the model has built in a noncommutative structure in phase space. The Hamiltonian is

$$\mathcal{H} = \sum_{j=0}^{\frac{N}{2}-1} \mathbf{P}_j \mathbf{X}^{j+1} \tag{34}$$

and the equations of motion for $\mathbf{X} \equiv \mathbf{X}^0$ gives $\frac{d^{N+1}}{dt^{N+1}}\mathbf{X}^0 = 0$. Note that in this case the order of derivatives, $(N + 1)$, is odd. The canonical generators of NGC algebra are

$$\begin{aligned}
H &= -\mathcal{H},\\
D &= -t\,\mathcal{H} + \sum_{j=0}^{\frac{N}{2}-1}(\frac{N}{2} - j)\frac{\{\mathbf{X}^j, \mathbf{P}_j\}_+}{2},\\
C &= -t^2\,\mathcal{H} + \sum_{j=0}^{\frac{N}{2}-1}\left((N - 2j)\,t\,\frac{\{\mathbf{X}^j, \mathbf{P}_j\}_+}{2} + (N - j + 1)j\,\mathbf{X}^{j-1}\,\mathbf{P}_j\right) + N(\frac{N}{2} + 1)\mathbf{P}_{\frac{N}{2}}\mathbf{X}^{\frac{N}{2}-1},\\
\boldsymbol{\mathcal{P}}_j &= j!\left(\sum_{\ell=0}^{j}\frac{t^{j-\ell}}{(j-\ell)!}\,\mathbf{P}_\ell\right), \qquad (j = 0, ..., \frac{N}{2} - 1),\\
\boldsymbol{\mathcal{P}}^b_j &= j!\left(\sum_{\ell=0}^{\frac{N}{2}}\frac{t^{j-\ell}}{(j-\ell)!}\,\mathbf{P}^b_\ell - M\sum_{\ell=\frac{N}{2}+1}^{j}(-)^\ell \times \frac{t^{j-\ell}}{(j-\ell)!}\,\mathbf{X}^{N-\ell}_a\epsilon^{ab}\right), \qquad (j = \frac{N}{2}, ..., N),\\
\mathbf{J}_{ab} &= \sum_{j=0}^{\frac{N}{2}}(\mathbf{X}^j_b\,\mathbf{P}_{ja} - \mathbf{X}^j_a\,\mathbf{P}_{jb}),\\
Z &= M.
\end{aligned} \tag{35}$$

where the central charge Z appears in

$$[\boldsymbol{\mathcal{P}}^a_j, \boldsymbol{\mathcal{P}}^b_k] = -i\,\epsilon^{ab}\,\delta^{N,j+k}\,(-1)^{\frac{j-k}{2}}k!\,j!\,Z. \tag{36}$$

As in the odd N case we can prove that these generators are constant of motion verifying the central extended NGC algebra. We can also prove that the associated Schrödinger equation is invariant under a scalar projective representation of the NGC group.

V. SUMMARY AND DISCUSSION

In this paper we have shown that the Schrödinger equation associated to the higher order non-relativistic particle is invariant under a projective representation of N(odd) GCA for any dimension. This results generalizes the well know result [3] [4] that the ordinary Schrödinger equation is invariant under the Schrödinger (conformal) group. In the case of 2+1 dimensions we have seen that the exotic Schrödinger equation is invariant under N(even) conformal algebras. We expect these results could be useful among other areas in the non-relativistic conformal condensed matter correspondence NGC /CMP.

The Hamiltonians (7) and (34) of present models are not positive semi-definite as is known in general for higher time derivative Lagrangian systems[4]. In quantum theory generically the system will contain ghost degrees of freedom. One possible procedure is to change the scalar product that has been applied to higher order harmonic oscillator [20] in reference [21]. Another possibility is to eliminate the ghost spectrum by imposing a BRST like operator on the physical states [22].

Possible extensions of the work is to consider the case of the NGC algebra for even N in any dimensions.We could also study the symmetry properties of the fourth order derivative harmonic oscillator [20], its generalizations. We expect in this case we will have a realization of the Newton-Hooke NGC algebra[14]. There are also possible higher order extensions of the Levy-Leblond equation [23] and the associated superconformal algebra [24].

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[4] A fourth order gravity in (2+1) dimensions without ghost has been studied in [19].

Appendix A: Particle Model action from Non-linear Realization

Here we show how the NGC invariant action (5), for odd N, is derived using the non-linear realization of the group G/H [15] (see also [26] and references there in), where G is the centrally extended algebra (3) with (11). The left invariant MC form $\Omega = -ig^{-1}dg$ is expanded as

$$\Omega = H\, L_H + D\, L_D + C\, L_C + \mathbf{P}_j\, L_{\mathbf{P}_j} + \frac{1}{2}\mathbf{J}_{ab}\, L^{ab}_{\mathbf{J}} + Z\; L_Z \tag{A1}$$

and is satisfying the MC equation $d\Omega + i\Omega\wedge\Omega = 0$. Using the NGC algebra (3) and (11) their components satisfy the MC equations,

$$\begin{aligned}
&dL_H = -L_D L_H, \qquad dL_D = -2L_C L_H, \qquad dL_C = L_D L_C,\\
&dL^a_{\mathbf{P}_j} = (j+1)\, L_H\, L^a_{\mathbf{P}_{j+1}} + (j-\frac{N}{2})\, L_D\, L^a_{\mathbf{P}_j} + (j-1-N)\, L_C\, L^a_{\mathbf{P}_{j-1}} - L^a_{\mathbf{J}b}L^b_{\mathbf{P}_j},\\
&dL^a_{\mathbf{J}b} = -L^a_{\mathbf{J}c}L^c_{\mathbf{J}b}, \quad dL_Z = \frac{1}{2}\sum_{j=0}^{N}\frac{(-1)^j}{{}_NC_j}\, L_{\mathbf{P}_j}\wedge L_{\mathbf{P}_{N-j}}.
\end{aligned}\tag{A2}$$

They are closed under "d" that is equivalent with that the Jacobi identities of the algebra are satisfied. The right hand side of dL_Z is the WZ two form closed and invariant in the non-extended algebra.

We parametrize the coset element as

$$g = e^{iHt}\, e^{i\mathbf{P}_j\tilde{\mathbf{X}}^j}\, e^{iC\sigma}\, e^{iD\rho}\, e^{iZc}\,. \tag{A3}$$

$(t, \tilde{\mathbf{X}}^0)$ for the generators $(H, \mathbf{P}_0)$ are identified to the $D = d + 1$ dimensional space-time coordinates. The left invariant MC form components are

$$L_H = e^{-\rho}\, dt, \qquad L_D = \,(d\rho - 2\sigma\, dt), \qquad L_C = e^{\rho}\,(d\sigma + \sigma^2\, dt), \tag{A4}$$

$$L_{\mathbf{P}_j} = e^{(j-\frac{N}{2})\rho}\sum_{k=0}^{j} {}_{N-k}C_{j-k}(-\sigma)^{j-k}\,(d\tilde{\mathbf{X}}^k - (k+1)\tilde{\mathbf{X}}^{k+1}\, dt),$$

$$L_Z = dc - \frac{\delta^{ab}}{2}\sum_{j=0}^{N}\left(\frac{(-1)^j}{{}_NC_j}\,\tilde{\mathbf{X}}^{N-j}_a\, d\tilde{\mathbf{X}}^j_b + \frac{(-1)^j j}{{}_NC_{j-1}}\,\tilde{\mathbf{X}}^{N-j+1}_a\,\tilde{\mathbf{X}}^j_b\, dt\right) \tag{A5}$$

Here and hereafter when there appear $\tilde{\mathbf{X}}^{-1}$ and $\tilde{\mathbf{X}}^{N+1}$, they are understood to be zero by definition. Note in the present parametrization of the coset L_Z does not depend on neither ρ nor σ.

In the method of NLR the particle action is constructed from $\mathbf{J}$ invariant one forms. They are L_H, L_D, L_C, L_Z and we can use their linear combination as the invariant action.

$$I=\int(\mathcal{L}_c+\mathcal{L}_{\mathbf{X}}),\quad \mathcal{L}_c=(b_H\,L_H+b_D\,L_D+b_C\,L_C)^*,\quad \mathcal{L}_{\mathbf{X}}=(a\,L_Z)^*, \tag{A6}$$

where * means pullback to the particle world line parametrized by τ. The first term $\mathcal{L}_c$ depends only on the su(1,1) variables, (t,σ,ρ), and giving one dimensional conformal mechanics Lagrangian [25].

We take the second term $\mathcal{L}_{\mathbf{X}}$ as the particle Lagrangian now depending on t and $\tilde{\mathbf{X}}^j$ in the present parametrization of the coset (A3). Using (A5)

$$\mathcal{L}_X=a\left[\dot{c}-\frac{\delta^{ab}}{2}\left(\sum_{j=0}^{N}\frac{(-1)^j}{{}_NC_j}\,\tilde{\mathbf{X}}_a^{N-j}\,\dot{\tilde{\mathbf{X}}}_b^j+\sum_{j=1}^{N}\frac{(-1)^j j}{{}_NC_{j-1}}\,\tilde{\mathbf{X}}_a^{N-j+1}\,\tilde{\mathbf{X}}_b^j\,\dot{t}\right)\right]d\tau \tag{A7}$$

and subtracting a surface term it becomes

$$\begin{aligned}\mathcal{L}_X=a&\left[\sum_{j=0}^{\frac{N-3}{2}}\frac{(-1)^j(j+1)}{{}_NC_j}\,\tilde{\mathbf{X}}^{N-j}\cdot\left(\tilde{\mathbf{X}}^{j+1}\dot{t}-\frac{1}{j+1}(\dot{\tilde{\mathbf{X}}}^j)\right)\right.\\&\left.+\frac{(-1)^{\frac{N-1}{2}}(\frac{N+1}{2}!)^2}{2N!\,\dot{t}}\left(\tilde{\mathbf{X}}^{\frac{N+1}{2}}\dot{t}-\frac{2}{N+1}\dot{\tilde{\mathbf{X}}}^{\frac{N-1}{2}}\right)^2-\frac{(-1)^{\frac{N-1}{2}}(\frac{N-1}{2}!)^2}{2N!\,\dot{t}}\left(\dot{\tilde{\mathbf{X}}}^{\frac{N-1}{2}}\right)^2\right]d\tau.\end{aligned} \tag{A8}$$

Here $\tilde{\mathbf{X}}^{N-j}$ in the first term runs over $\tilde{\mathbf{X}}^{\frac{N+3}{2}},...,\tilde{\mathbf{X}}^N$ and they play roles of Lagrange multipliers giving their equations of motion ,

$$\tilde{\mathbf{X}}^{j+1}=\frac{1}{j+1}(\frac{\dot{\tilde{\mathbf{X}}}^j}{\dot{t}}),\qquad j=0,...,\frac{N-3}{2}. \tag{A9}$$

$\tilde{\mathbf{X}}^{\frac{N+1}{2}}$ equation of motion determines $\tilde{\mathbf{X}}^{\frac{N+1}{2}}$ as

$$\tilde{\mathbf{X}}^{\frac{N+1}{2}}=\frac{2}{N+1}(\frac{\dot{\tilde{\mathbf{X}}}^{\frac{N-1}{2}}}{\dot{t}}). \tag{A10}$$

Using it back into the Lagrangian (A8) the second term can be dropped. $\tilde{\mathbf{X}}^{\frac{N+1}{2}},...,\tilde{\mathbf{X}}^1$ are solved iteratively in terms of $\tilde{\mathbf{X}}^0$ and its derivatives,

$$\tilde{\mathbf{X}}^j=\frac{1}{j!}\,D^j\tilde{\mathbf{X}}^0,\qquad D\equiv\frac{1}{\dot{t}}\frac{d}{d\tau},\qquad j=1,...,\frac{N+1}{2}. \tag{A11}$$

If we use them back in the Lagrangian we obtain

$$\mathcal{L}_X=a\,\frac{(-1)^{\frac{N+1}{2}}}{2N!}(D^{\frac{N+1}{2}}\tilde{\mathbf{X}}^0)^2\dot{t}\,d\tau. \tag{A12}$$

In a static gauge $\dot{t} = 1$ the Lagrangians (A8) and (A12) become (5) and (2) respectively by identifications $a = (-1)^{\frac{N+1}{2}} N!\, M$ and

$$\tilde{\mathbf{X}}^j = \frac{\mathbf{X}^j}{j!},\ (j = 0, ..., \frac{N-1}{2}), \quad \tilde{\mathbf{X}}^j = \frac{(-1)^j\, N!}{a\, j!} \mathbf{Y}^{N-j},\ (j = \frac{N+3}{2}, ..., N). \tag{A13}$$

For even N case the central extension is possible only in $D = 1 + 2$ dimensions (36). Applying the similar discussions we arrive at the actions (29) and (30).

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